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Adjusted Control Limits for P Charts

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Abstract: P charts are used for count data following the Binomial distribution. However, the P chart has symmetrical control limits when the Binomial distribution is nonsymmetrical. As a result, the upper control limit can have a rate of false detection as high as 1 in 11.5 points plotted. This can result in wasted resources investigating false signals. Further, the lower control limit has a rate of false detection consistently above 1 in 1000. This makes it slow to detect improvements in quality. Adjusted control limits are provided that correct both these problems. The adjusted control limits result in a P chart truly based on the assumption of the Binomial distribution.

1.0 Introduction

The average run length (ARL) of a control chart is the average number of points that are plotted before one goes outside the control limits. The ARL varies based on the size of the shift. When a significant shift occurs the ARL should be small, approaching 1. Also of interest is the ARL when there is no shift. This is the time between false signals. It should be large. The ARL when there is no change will be referred to as the false detection time, FDT. $1/\text{FDT}$ will be referred to as the false detection rate, FDR.

Control charts based on the normal distribution, such as \bar{X} and I_N charts (Taylor 2017b), with ± 3 standard deviation control limits, have FDRs of:

$$1 \text{ in } 740 \text{ relative to the upper control limit: } 1 - \Phi(3) = 0.0013499 = 1/740.80$$

$$1 \text{ in } 740 \text{ relative to the lower control limit: } \Phi(-3) = 0.0013499 = 1/740.80$$

$$1 \text{ in } 370 \text{ for both control limits combined: } 2 \times 0.0013499 = 1/370.40$$

$\Phi(z)$ is the probability of being less than z for the normal distribution ($\mu=0, \sigma=1$).

P charts assume pass/fail data and are based on the Binomial distribution. The Binomial distribution has 2 parameters: sample size (n) and probability fail (p), where:

$$\text{Average} = np \qquad \text{Standard Deviation} = \sqrt{np(1-p)}$$

The standard control limits for a P chart are:

$$\text{LCL}_{\text{Standard}} = np - 3\sqrt{np(1-p)} \qquad \text{UCL}_{\text{Standard}} = np + 3\sqrt{np(1-p)}$$

As the Binomial distribution is not symmetrical and the standard control limits are symmetrical, the FDRs of a P chart will differ from those above.

3 standard deviation control limits are generally robust to the assumption of normality as most distributions have a high percentage of values within three standard deviations of the average. While this may be true most of the time, for the P chart the false detection rate can be as high as 1 in 11.5. This is an unacceptable rate. This article documents the false detection rates for P charts and offers a solution in the form of an adjustment to the control limits.

2.0 False Detection Rates

The P chart works differently than an \bar{X} chart when detecting a worsening of quality. For an \bar{X} chart, both upward and downward shifts can signal a shift from the target and a worsening of quality. The FDR relative to a worsening in quality is then 1 in 370.

For a P chart of failures or nonconforming units, only an increase in the counts signals a worsening of quality. Therefore, it is appropriate to have a 1 in 370 FDR associated with the UCL by itself. It would be of concern if the FDR dropped below 1 in 200, as this represents nearly a doubling in the number of false signals.

The lower control limit signals an improvement in quality. The consequence of a false detection relative to the LCL is different from that associated with the UCL. It is also appropriate to have a 1 in 370 FDR associated with the LCL by itself. It would be of concern if the FDR dropped below 1 in 200.

The article Adjusted Control Limits for U Charts, Taylor (2017a), describes how the FDR for U charts can be as high as 1 in 11.5. As shown below, the same is true for P charts. This is an unacceptable rate. This article determines when the standard control limits for a P chart can be safely used, i.e., the FDR is at worst 1 in 200. This article also shows how to handle situations where the FDR is worse than 1 in 200.

Figures 1-5 show the FDR as a function of the average count ($n \times p$). Formulas for the FDR are given in Section 4. The FDR is not a smooth curve. This is due to counts being integers. The FDR jumps whenever the control limits cross an integer value. This creates an oscillation pattern.

As long as $10 \leq np \leq n-10$, the FDR is better than 1 in 200. This is true for all sample sizes, even the minimum sample size satisfying these conditions of $n=20$. When n is 100 or more, the FDR is close to the desired 1 in 370 over a wide range. P charts with standard control limits should only be used when $10 \leq np \leq n-10$. This corresponds to the conditions where the Binomial distribution is well approximated by the normal distribution.

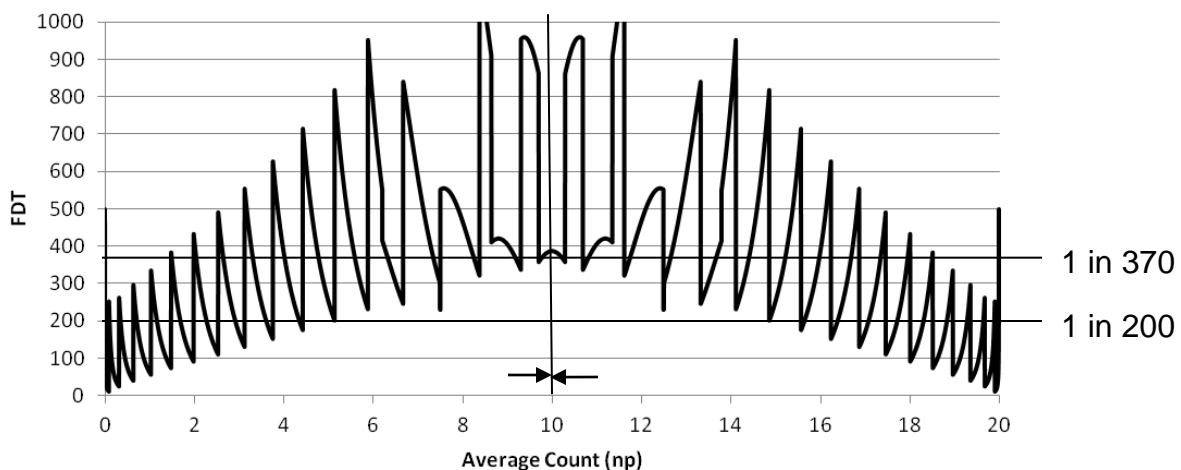


Figure 1: False Detection Rate for the Standard Control Limits, $n=20$

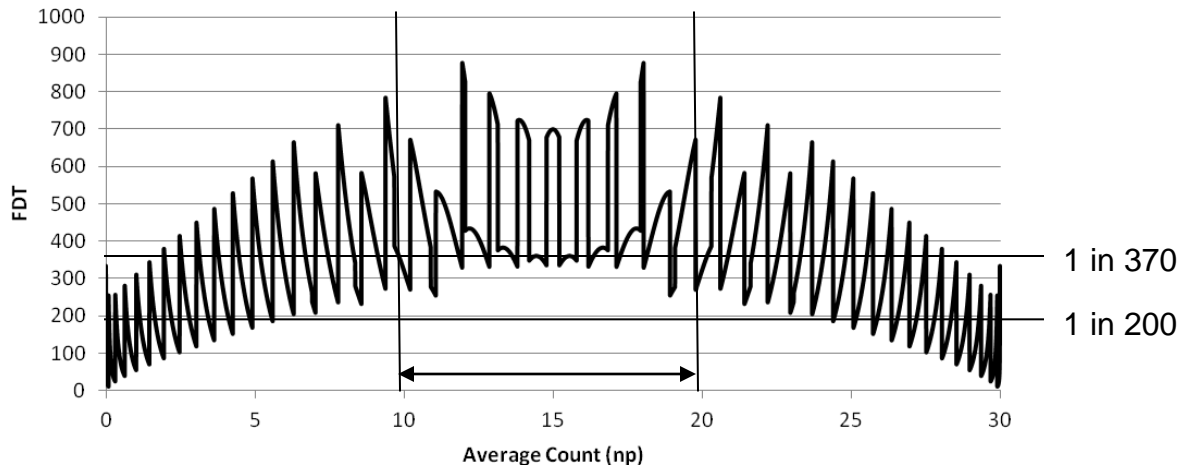


Figure 2: False Detection Rate for the Standard Control Limits, n=30

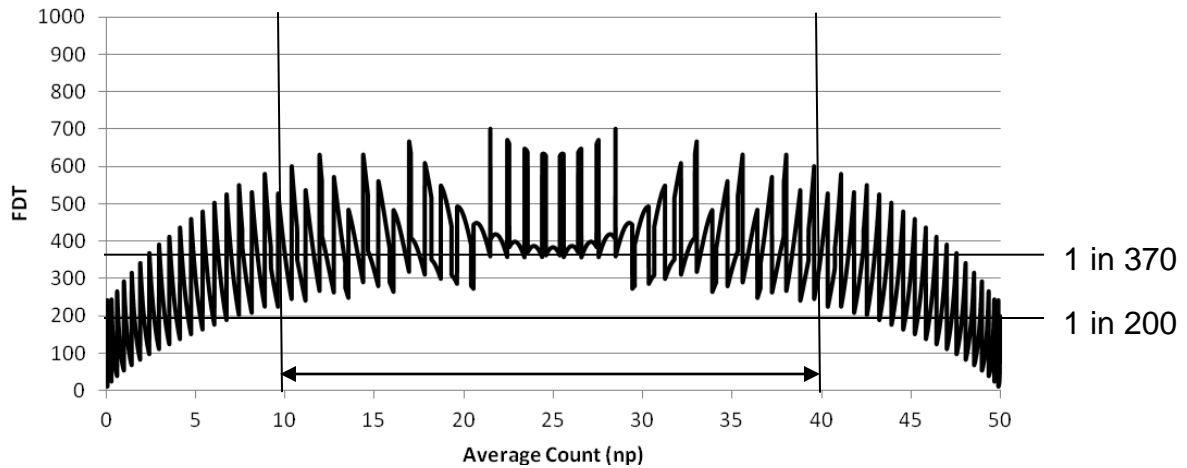


Figure 3: False Detection Rate for the Standard Control Limits, n=50

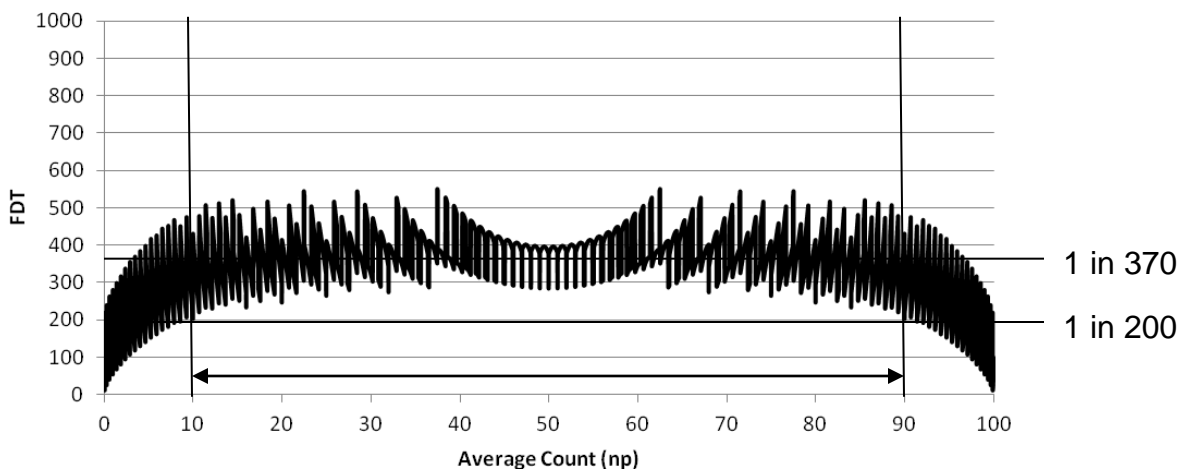


Figure 4: False Detection Rate for the Standard Control Limits, n=100

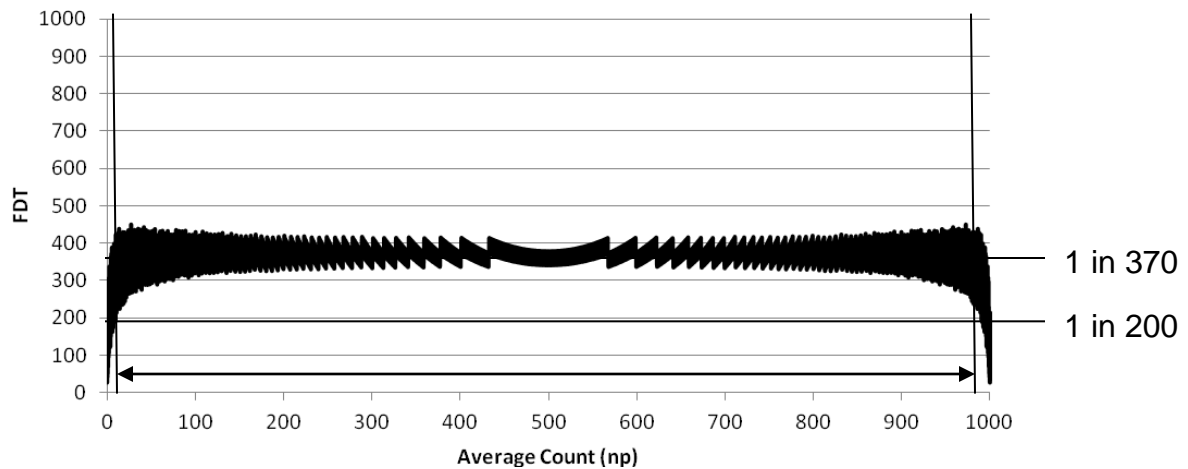


Figure 5: False Detection Rate for the Standard Control Limits, n=1000

3.0 Adjusted Control Limits

It is common for P charts to have an average count $np < 10$. STAT-10 Statistical Techniques for Trending Data in Taylor (2017c) states the P chart is generally the best chart for pass/fail counts less than 25 but that the I_N chart (or Laney P' chart) is generally the best chart for counts greater than 25. The niche for the P chart is when counts are less than 25. This includes counts < 10 where adjusted control limits are needed.

One approach is to cumulate data over longer periods of time. For example, instead of performing a weekly chart with an average count of 2.5, perform a monthly chart with an average count of 10. While this may better control the false rejection rate, it may also delay the detection of a problem.

Another approach takes advantage of the relationship between the binomial and Poisson distributions. Holding the average count constant, as n gets larger the binomial distribution converges to the Poisson distribution with $\lambda = np$. This means the adjusted control limits for the U chart in Taylor (2017a) may be used with a P chart. The adjusted control limits for the P chart are then:

$$UCL_{Adjusted} = np + 2.782\sqrt{np(1-p)} + 1$$

$$LCL_{Adjusted} = np - 2.782\sqrt{np(1-p)} + 1.1$$

These are the adjusted control limits for the U chart with the standard deviation of the binomial distribution substituted for the standard deviation of the Poisson distribution.

Figures 6-9 show the FDR of the adjusted control limits as a function of the average count (np). The FDR is calculated based on the binomial distribution, so is exact. In all the charts the FDR is better than 1 in 200. However, for $n=50$ the FDR diverges from 1 in 370. For $n=100$ and above, the FDR stays in the 1 in 370 region. For $n \geq 100$ and $np < 10$, the adjusted control limits may be used.

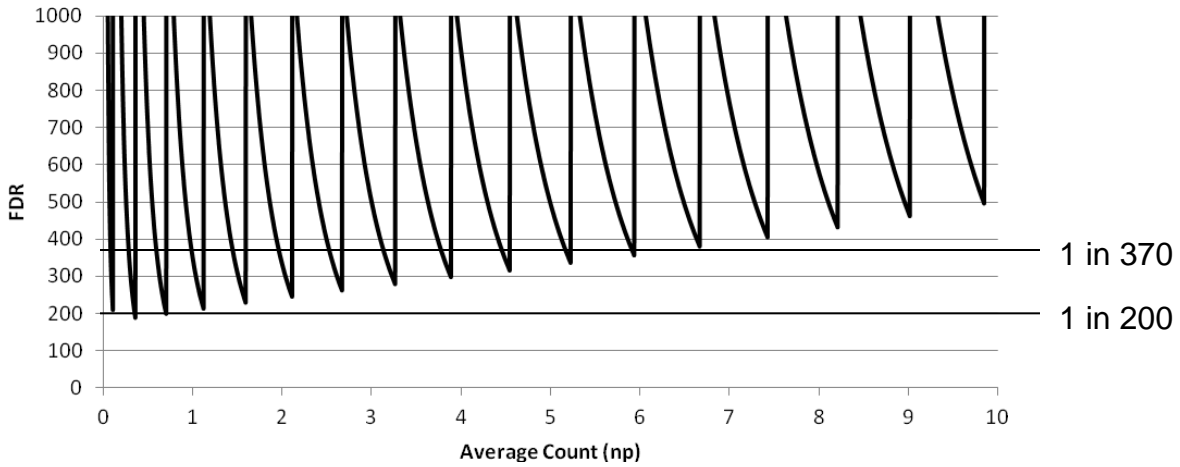


Figure 6: False Detection Rate for the Adjusted Upper Control Limit, n=30

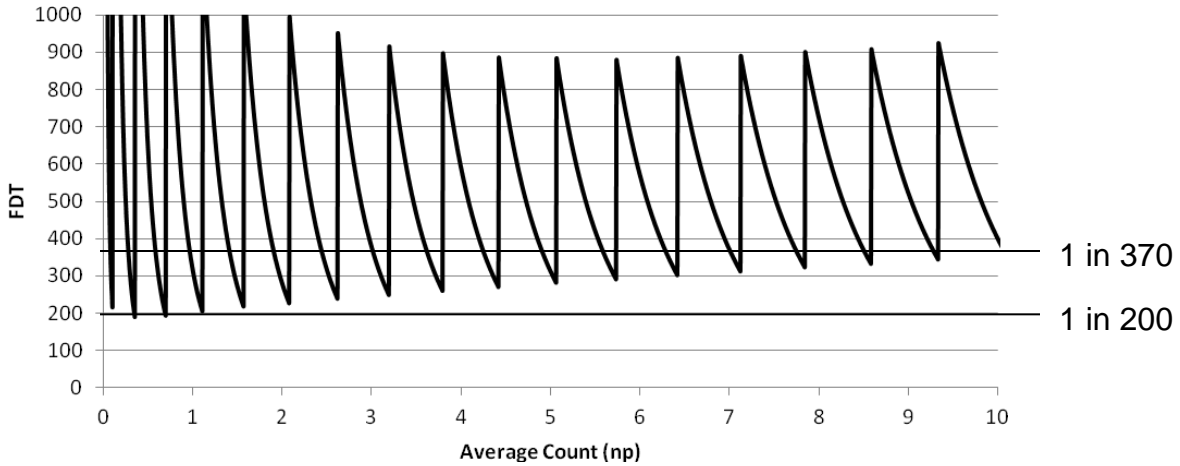


Figure 7: False Detection Rate for the Adjusted Upper Control Limit, n=50

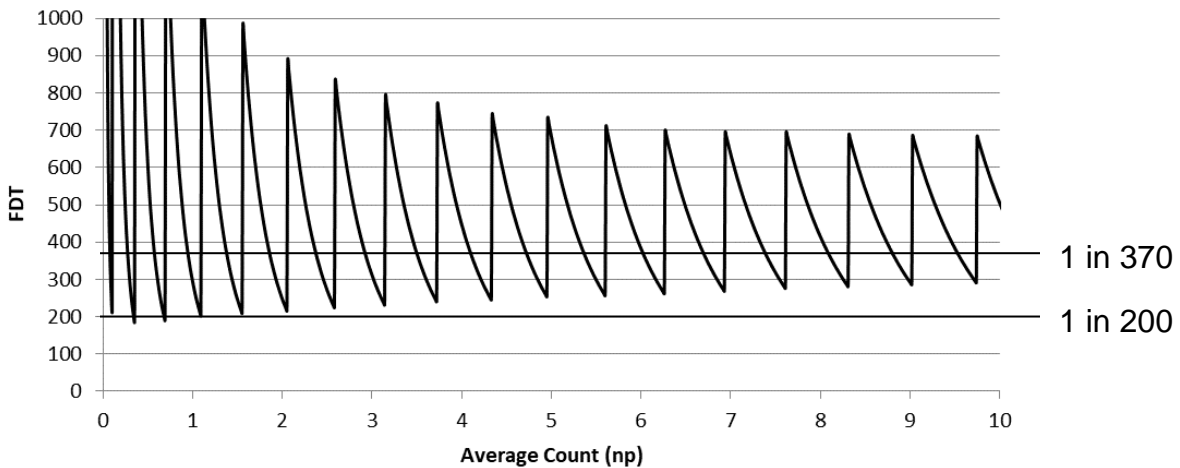


Figure 8: False Detection Rate for the Upper Control Limit, n=100

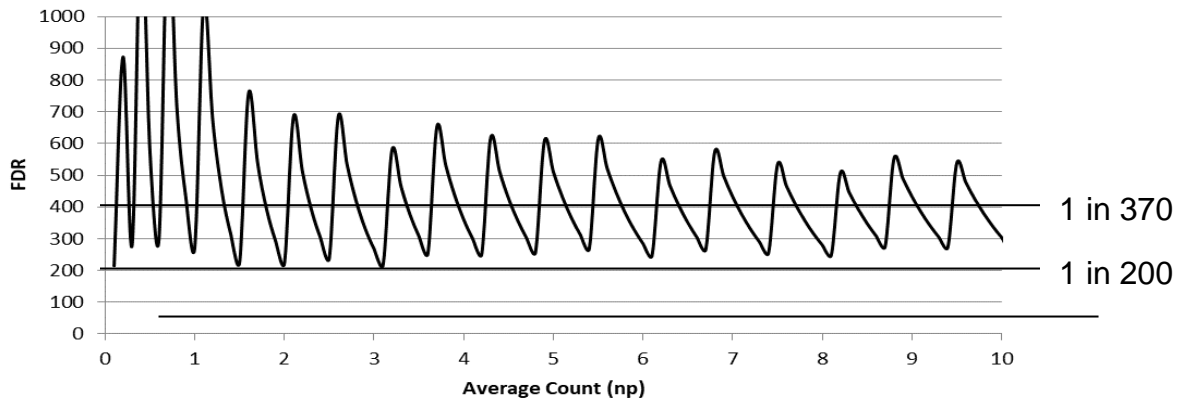


Figure 9: False Detection Rate for the Upper Control Limit, n=1000

For $n \geq 100$ and $np > n - 10$, using symmetry the adjusted control limits below can be used:

$$LCL_{Adjusted} = np - 2.782\sqrt{np(1-p)} - 1$$

$$UCL_{Adjusted} = np + 2.782\sqrt{np(1-p)} - 1.1$$

This only leaves the situation where $n < 100$ and either $np < 10$ or $np > n - 10$. In this case cumulate data until there are at least 100 samples.

4.0 P Chart Formulas

In the previous sections it was assumed that each point is a count F_i following the binomial distribution with parameters n and p :

$$F_i \sim \text{Binomial}(n, p)$$

The binomial distribution has the following properties:

$$\text{Average} = np$$

$$\text{Standard Deviation} = \sqrt{np(1-p)}$$

The fact that p is used to estimate the standard deviation simplifies the chart as a separate estimate of the standard deviation is not needed and makes the chart more powerful for binomial data. However, it makes the chart sensitive to the assumption of the binomial distribution. The binomial distribution occurs when the items being counted occur independently of each other. When counts are larger, the counts tend to vary more than the binomial due to dependencies. This is called overdispersed. Nonconforming units may occur in runs, occur because of a problem with a single cavity or nozzle, etc.

For the binomial distribution, the number of failing units F_i are plotted along with the following control limits:

$$LCL_{Counts} = np - 3\sqrt{np(1-p)}$$

$$UCL_{Standard} = np + 3\sqrt{np(1-p)}$$

LCL=0 whenever the lower control limit is negative. UCL=n whenever the upper control limit is greater than n. The adjusted control limits for $n \geq 100$ and $np < 10$ are:

$$UCL_{Adjusted} = np + 2.782\sqrt{np(1-p)} + 1$$

$$LCL_{Adjusted} = np - 2.782\sqrt{np(1-p)} + 1.1$$

$$=0 \text{ for } p \leq \frac{2.77024877318692 + \sqrt{6.46427826534363 - 9.36600203111234/n}}{n + 7.74049754637384}$$

This was obtained by setting the LCL to zero and solving for p.

The value $2.78217496688721 = -\Phi^{-1}(2\Phi(-3))$ was selected to have the same FDR for just the UCL as for the standard deviation control limits. This is around 1 in 370.

The adjusted control limits for $n \geq 100$ and $np > n-10$ are:

$$UCL_{Adjusted} = np + 2.782\sqrt{np(1-p)} - 1.1$$

$$=n \text{ for } p \geq 1 - \frac{2.77024877318692 + \sqrt{6.46427826534363 - 9.36600203111234/n}}{n + 7.74049754637384}$$

$$LCL_{Adjusted} = np - 2.782\sqrt{np(1-p)} - 1$$

The false detection rates are:

$$FDR_{Lower} = \frac{1}{\text{Binomial}[\text{ceil}(LCL) - 1 | n, p]}$$

$$FDR_{Upper} = \frac{1}{1 - \text{Binomial}[\text{floor}(UCL) | n, p]}$$

$$FDR_{Both} = \frac{1}{\text{Binomial}[\text{ceil}(LCL) - 1 | n, p] + (1 - \text{Binomial}[\text{floor}(UCL) | n, p])}$$

Ceil() rounds the value up to an integer. Floor() rounds the value down to an integer. Binomial() is the binomial distribution function.

A control chart of counts is referred to as an NP chart. For a P chart, proportions P_i are plotted based on the sample sizes N_i :

$$P_i = \frac{F_i}{N_i} \text{ with } F_i \sim \text{Binomial}(N_i, p)$$

Then the control limits for the proportions P_i are:

$$UCL_{Standard} = \frac{N_i p + 3\sqrt{N_i p(1-p)}}{N_i} = p + 3\sqrt{\frac{p(1-p)}{N_i}}$$

$$LCL_{Standard} = \frac{N_i p - 3\sqrt{N_i p(1-p)}}{N_i} = p - 3\sqrt{\frac{p(1-p)}{N_i}}$$

LCL=0 whenever the lower control limit is negative. UCL=1 whenever the upper control limit is greater than 1. Similarly, the adjusted control limits for $n \geq 100$ and $np < 10$ are:

$$UCL_{Adjusted} = p + 2.782\sqrt{\frac{p(1-p)}{N_i}} + \frac{1}{N_i}$$

$$LCL_{Adjusted} = p - 2.782\sqrt{\frac{p(1-p)}{N_i}} + \frac{1.1}{N_i}$$

$$= 0 \text{ for } p \leq \frac{2.77024877318692 + \sqrt{6.46427826534363 - 9.36600203111234 / N_i}}{(N_i + 7.74049754637384)}$$

The adjusted control limits for $n \geq 100$ and $np > n-10$ are:

$$UCL_{Adjusted} = p + 2.782\sqrt{\frac{p(1-p)}{N_i}} - \frac{1.1}{N_i}$$

$$= 1 \text{ for } p \geq 1 - \frac{2.77024877318692 + \sqrt{6.46427826534363 - 9.36600203111234 / N_i}}{(N_i + 7.74049754637384)}$$

$$LCL_{Adjusted} = p - 2.782\sqrt{\frac{p(1-p)}{N_i}} - \frac{1}{N_i}$$

5.0 Conclusions

P charts are useful for pass/fail count data following the Binomial distribution, which is most likely for counts below 25. For the Binomial distribution, the P chart is better than an I_n chart because it provides a better estimate of the standard deviation. However, a P chart is not entirely based on the Binomial distribution because the Binomial distribution is skewed while a P chart uses symmetrical control limits.

For the standard upper control limit, the rate of false detection can be as high as 1 in 11.5 for low counts. It does not remain above 1 in 200 until the average count reaches 10 or is less than $n-10$.

The adjusted UCL control limit fixes this problem for $n \geq 100$. It maintains the false detection rate above 1 in 200 and averages around 1 in 370. For $n < 100$ and either

$np < 10$ or $np > n - 10$, cumulate data until there are at least 100 samples. The adjusted UCL also improves the detection of a worsening of quality for larger counts, as long as the assumption of the Poisson continues to hold.

For the standard lower control limit, the rate of false detection is consistently above 1 in 1000. This makes it slow to detect improvements in quality. This can result in missing improvements and the associated lessons learned. The adjusted LCL makes the chart much quicker at detecting improvements in quality.

The adjusted control limits result in a P chart truly based on the assumption of the Binomial distribution. The same approach can be applied to U charts as described in Taylor (2017a).

6.0 References

Adjusted Control Limits for U Charts (2017a), Dr Wayne A. Taylor, Taylor Enterprises, Inc. (www.variation.com/techlib/brief2.html).

Normalized Individuals (I_N) Chart (2017b), Dr Wayne A. Taylor, Taylor Enterprises, Inc. (www.variation.com/techlib/brief1.html).

Statistical Procedures for the Medical Device Industry (2017c), Dr Wayne A. Taylor, Taylor Enterprises, Inc. (<http://www.variation.com/procedures>).