Adjusted Control Limits for U Charts

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Abstract: U charts are used for count data following the Poisson distribution. However, the U chart has symmetrical control limits when the Poisson distribution is nonsymmetrical. As a result, the upper control limit can have a rate of false detection as high as 1 in 11.5 points plotted. This can result in wasted resources investigating false signals. Further, the lower control limit has a rate of false detection consistently above 1 in 1000. This makes it slow to detect improvements in quality. Adjusted control limits are provided that correct both these problems. The adjusted control limits result in a U chart truly based on the assumption of the Poisson distribution.

1.0 Introduction

The average run length (ARL) of a control chart is the average number of points that are plotted before one goes outside the control limits. The ARL varies based on the size of the shift. When a significant shift occurs the ARL should be small, approaching 1. Also of interest is the ARL when there is no shift. This is the time between false signals. It should be large. The ARL when there is no change will be referred to as the false detection time, FDT. 1/FDT will be referred to as the false detection rate, FDR.

Control charts based on the normal distribution, such as $\bar{X}$ and I_N charts (Taylor 2017b), with ±3 standard deviation control limits, have FDRs of:

1 in 740 relative to the upper control limit: $1-\Phi(3)=0.0013499=1/740.80$

1 in 740 relative to the lower control limit: $\Phi(-3)=0.0013499=1/740.80$

1 in 370 for both control limits combined: $2\times 0.0013499=1/370.40$

$\Phi(z)$ is the probability of being less than z for the normal distribution (μ=0, σ=1).

U charts assume count data and are based on the Poisson distribution. The Poisson distribution has one parameter, the average count λ. For the Poisson distribution:

$$\text{Average}=\lambda \qquad \text{Standard Deviation}=\sqrt{\lambda}$$

The standard control limits for a U chart are:

$$LCL_{Standard}=\text{Average}-3\sqrt{\text{Average}} \qquad UCL_{Standard}=\text{Average}+3\sqrt{\text{Average}}$$

As the Poisson distribution is not symmetrical and the standard control limits are symmetrical, the FDRs of a U chart will differ from those above.

3 standard deviation control limits are generally robust to the assumption of normality as most distributions have a high percentage of values within three standard deviations of the average. While this may be true most of the time, for the U chart the false detection rate can be as high as 1 in 11.5. This is an unacceptable rate. This article documents the false detection rates for U charts and offers a solution in the form of an adjustment to the control limits.

2.0 False Detection Rates

The U chart works differently than an $\bar{X}$ chart when detecting a worsening of quality. For an $\bar{X}$ chart, both upward and downward shifts can signal a shift from the target and a worsening of quality. The FDR relative to a worsening in quality is then 1 in 370.

For a U chart of complaints or nonconformities, only an increase in the counts signals a worsening of quality. Therefore, it is appropriate to have a 1 in 370 FDR associated with the UCL by itself. It would be of concern if the FDR dropped below 1 in 200, as this represents nearly a doubling in the number of false signals.

The lower control limit signals an improvement in quality. The consequence of a false detection relative to the LCL is different from that associated with the UCL. It is also appropriate to have a 1 in 370 FDR associated with the LCL by itself. It would be of concern if the FDR dropped below 1 in 200.

Figure 1 shows the FDR for the UCL as a function of the average count for two different scales. Formulas for the FDR are given in Section 4. The FDR is not a smooth curve due to counts being integers. The FDR jumps whenever the control limit crosses an integer value. This creates an oscillation pattern. The FDR can be as low as 11.5. It does not stay above 1 in 200 until the average count reaches 10. Standard control limits should not be used due to false rejections when the average is less than 10.

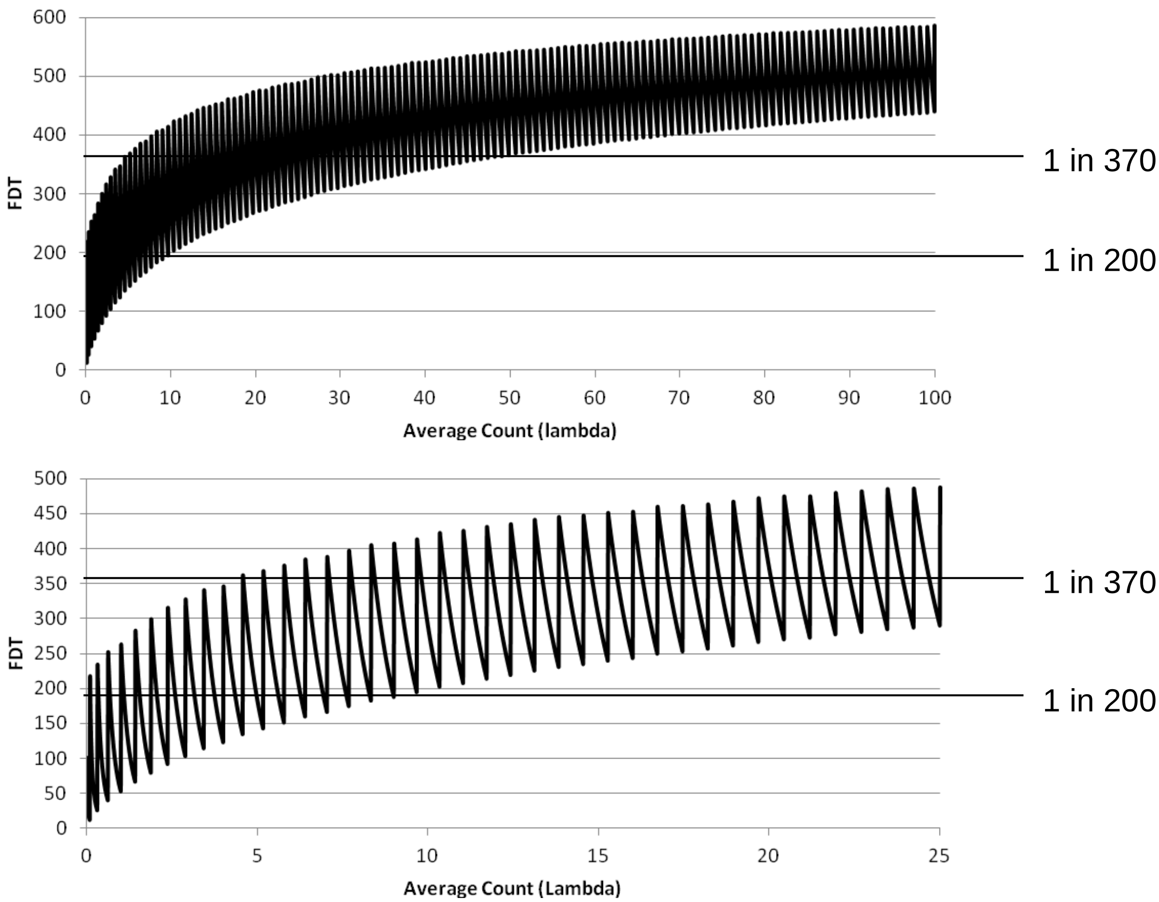


Figure 1: False Detection Rate for the Standard Upper Control Limit

As the average count exceeds 70, the FDR for the UCL varies between 1 in 400 and 1 in 600. This is less frequent than the target 1 in 370. This means the U chart is not as effective as it could be at detecting a worsening in quality.

Figure 2 shows the FDR associated with the LCL as a function of the average count. The LCL is zero until the average count reaches 9. This means the FDR is infinite. Above 9 the FDR is consistently above 1 in 1000 and generally far above this. This means the LCL is not as effective as it could be at detecting an improvement in quality.

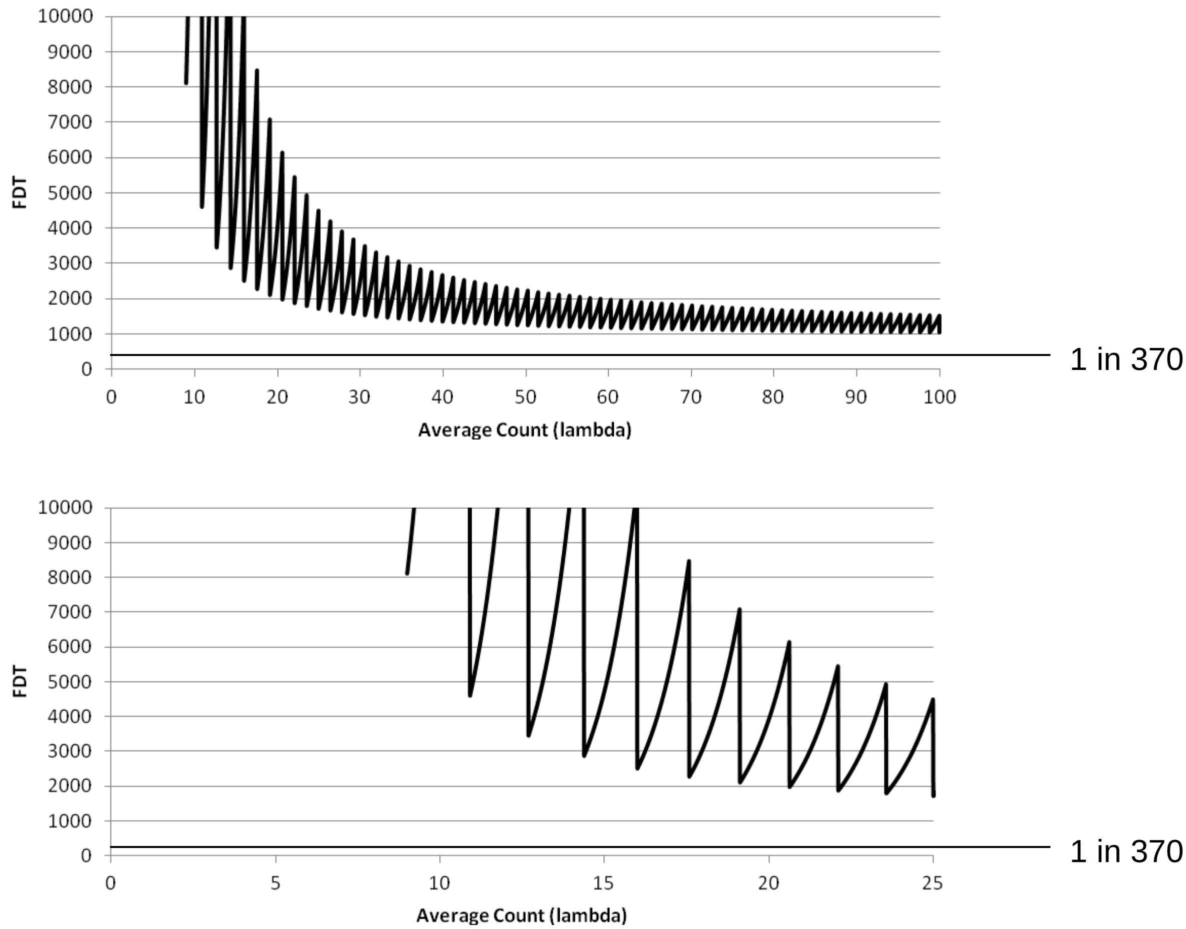


Figure 2: False Detection Rate for the Standard Lower Control Limit

As the counts increase the Poisson distribution more closely resembles the Normal distribution. At the same time, the counts are less likely to follow the Poisson distribution. STAT-10 Statistical Techniques for Trending Data in Taylor (2017c) states the U chart is generally the best chart for counts less than 25 but that the I_N chart (or Laney U' chart) is generally the best chart for counts greater than 25. For counts greater than 25 the data tends to be normal but overdispersed, meaning it varies more than the Poisson distribution.

The niche for the U chart is when counts are less than 25. This is when the counts are most likely to follow the Poisson distribution. It is also when the counts are most skewed, so most nonnormal. The U chart is based on Poisson property that the standard deviation is the square root of the average. But U charts also use symmetrical control limits when the Poisson distribution is not symmetrical. This is why adjusted control limits are needed.

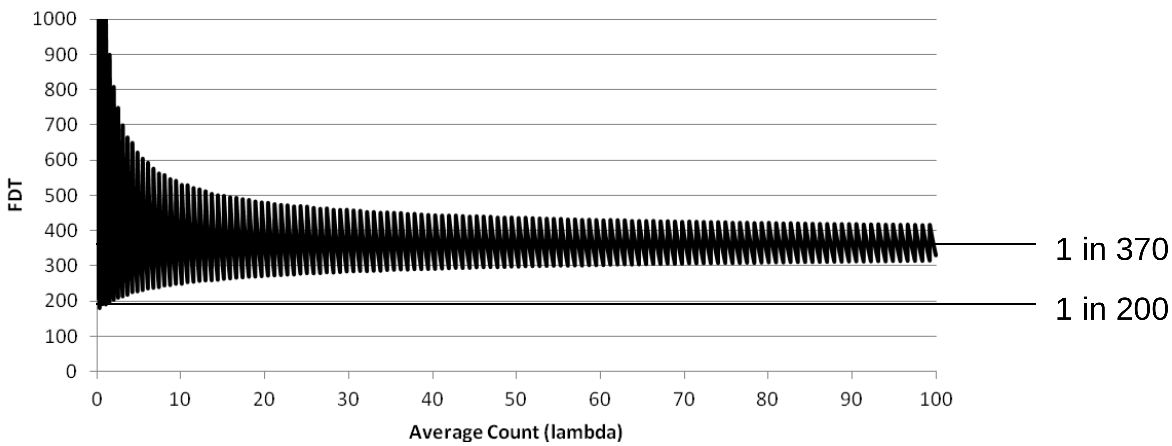
3.0 Adjusted Control Limits

The formula for the adjusted UCL is:

$$UCL_{Adjusted} = Average + 2.782\sqrt{Average} + 1$$

2.782 is from the Normal distribution. It corresponds to a 1 in 370 FDR just like 3 corresponds to 1 in 740 FDR. The adjustment factor 1, adjusts for both the discreteness of the values and positive skewness. It was selected so that the minimum FDR is near 1 in 200, as shown in Figure 3.

The adjustment ($UCL_{Adjusted}$ - $UCL_{Standard}$) increases the UCL by slightly less than 1 when the average is 1 or below, reducing the false detection rate. For larger counts, the correction lowers the UCL by up to 1, making the chart quicker to react.



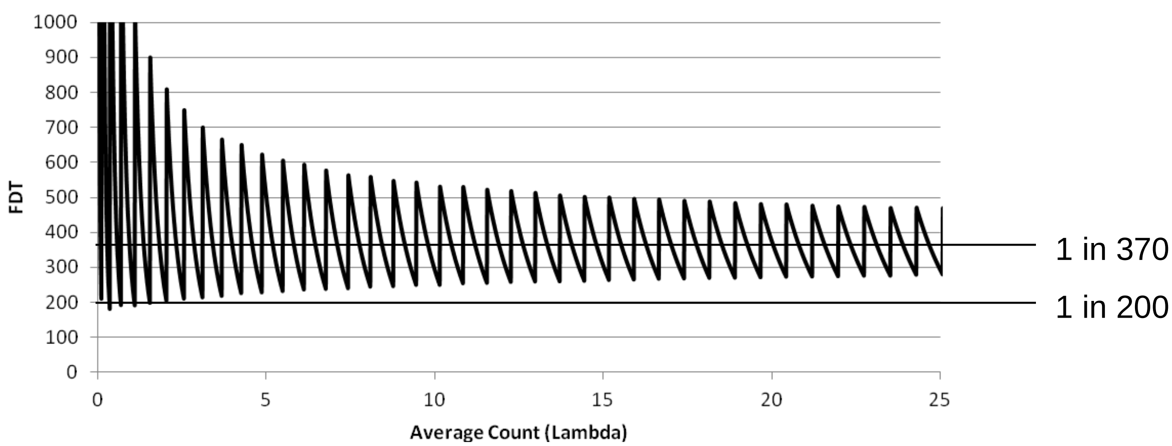


Figure 3: False Detection Rate for the Adjusted Upper Control Limit

The formula for the adjusted LCL is:

$$LCL_{Adjusted} = Average - 2.782\sqrt{Average} + 1.1$$

The formula is valid for averages of 5.313 and above. Below that the LCL is zero. As before, 2.782 is from the Normal distribution. The adjustment factor 1.1 adjusts for both the discreteness of the values and positive skewness. It was selected so that the minimum FDR is near 1 in 200 (Figure 4).

The adjustment ($UCL_{Adjusted}$ - $UCL_{Standard}$) increases the LCL by 1.1 for smaller counts and up to 3.3 for higher counts, making the chart quicker to react to an improvement in quality.

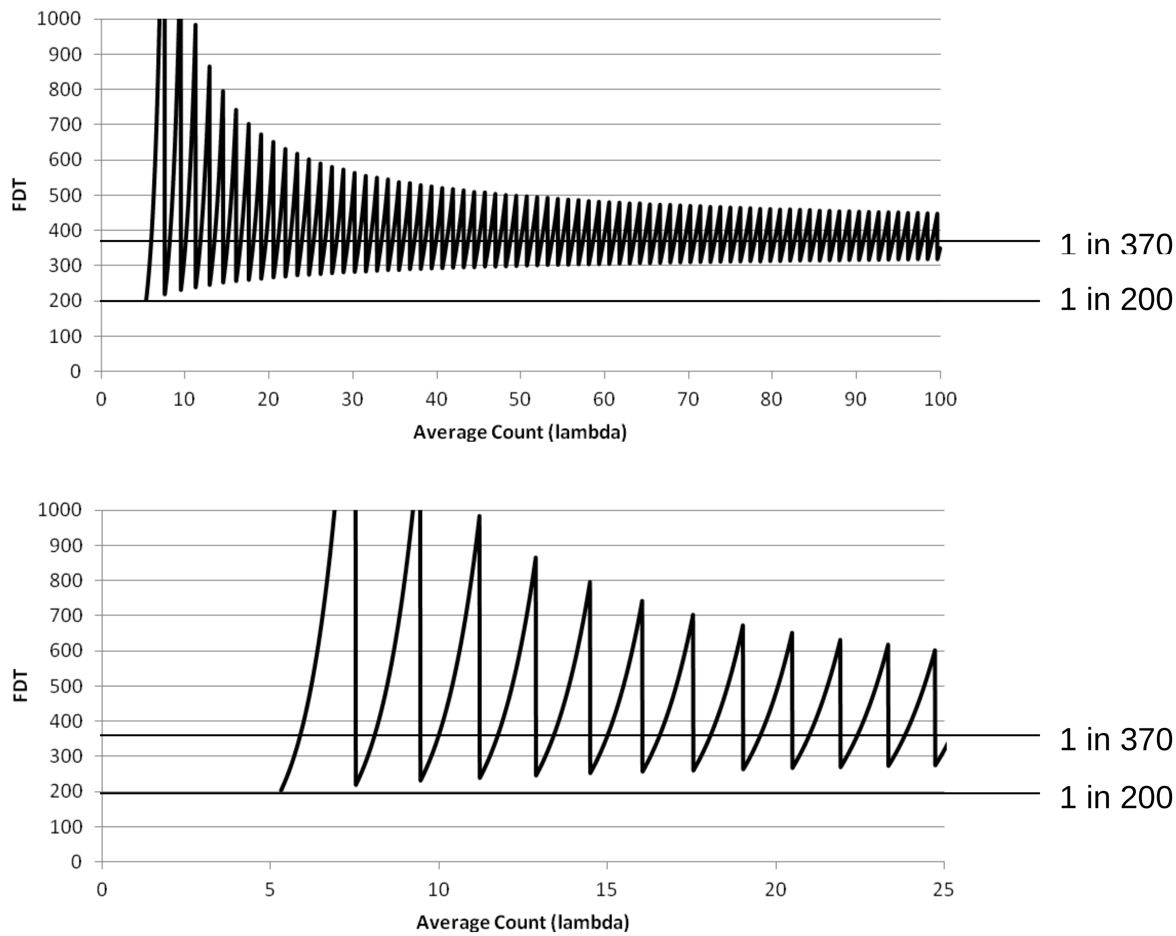


Figure 4: False Detection Rate for the Adjusted Lower Control Limit

An alternative approach is to use what is sometimes called exact control limits. It involves setting the control limits as:

$$UCL_{Exact} = \min x \text{ where 1-Poisson}\left[x \middle| Average\right] \le \frac{1}{740}$$

$$LCL_{Exact} = \max x \text{ where Poisson}\left[x \middle| \text{Average}\right] \le \frac{1}{740}$$

This results in the curves like those in Figures 3 and 4 being entirely above 740. It would be better to use 1/370 than 1/740 for the reasons given before. While exact control limits resolve the false detection issue, they are not as fast at detecting a change compared to the adjusted control limits.

4.0 U Chart Formulas

It is assumed that each point is a count C_i following the Poisson distribution with parameter λ:

$$C_i \sim \text{Poisson}(\lambda)$$

The Poisson distribution has the property:

$$\text{Standard Deviation} = \sqrt{\text{Average}} \text{ where Average} = \lambda$$

The fact that the average is used to estimate the standard deviation simplifies the chart as a separate estimate of the standard deviation is not needed and makes the chart more powerful for Poisson data. However, it makes the chart sensitive to the assumption of the Poisson distribution. The Poisson distribution occurs when the items being counted occur independently of each other. When counts are larger, the counts tend to vary more than the Poisson due to dependencies. This is called overdispersed. For example, complaints may be grouped together for processing or multiple particles may be introduced at the same time.

For counts the standard control limits are:

$$UCL_{Standard} = \text{Average} + 3\sqrt{\text{Average}}$$

$$LCL_{Standard} = \text{Average} - 3\sqrt{\text{Average}}$$

$LCL_{Standard}$=0 for $\lambda \le 9$. This assumes the number of opportunities or sample size is constant. This is a special case of a U chart that is commonly called a C chart.

The adjusted control limits are:

$$UCL_{Adjusted} = \text{Average} + 2.782\sqrt{\text{Average} + 1}$$

$$LCL_{Adjusted} = \text{Average} - 2.782\sqrt{\text{Average} + 1.1}$$

$$= 0 \text{ for } \lambda \le 5.313 = \left(-\Phi^{-1}\left(2\Phi(-3)\right) \pm \sqrt{\Phi^{-1}\left(2\Phi(-3)\right)^2 - 4.4}\right)^2 \frac{1}{4}.$$

The value 2.782 is more exactly $-\Phi^{-1}\left(2\Phi(-3)\right)$ = 2.78217496688721. It was selected to give an FDR for just the UCL equal to that of both control limits for the $\overline{X}$ and I_N charts. This is around 1 in 370. As the average count increases, the FDR relative to the UCL converges to around 1 in 370 as shown in Figures 3 and 4.

The false detection rates are:

$$FDR_{Lower} = \frac{1}{Poisson\left[ceil\left(LCL\right)-1\middle|\lambda\right]}$$

$$FDR_{Upper} = \frac{1}{1\text{-}Poisson\left[floor\left(UCL\right)\middle|\lambda\right]}$$

$$FDR_{Both} = \frac{1}{Poisson\left[ceil\left(LCL\right)-1\middle|\lambda\right]+\left(1\text{-}Poisson\left[floor\left(UCL\right)\middle|\lambda\right]\right)}$$

Ceil() rounds the value up to an integer. Floor() rounds the value down to an integer. Poisson() is the Poisson distribution function.

A control chart of counts is referred to as a C chart. For a U chart, rates R_i are plotted based on the number of opportunities O_i:

$$R_i = \frac{C_i}{O_i} \quad \text{with} \qquad C_i \sim Poisson\left(\lambda O_i\right)$$

Then the standard control limits for the rates R_i are:

$$UCL_{Rates\text{-}Standard} = \frac{\lambda O_i + 3\sqrt{\lambda O_i}}{O_i} = \lambda + 3\sqrt{\frac{\lambda}{O_i}}$$

$$LCL_{Rates\text{-}Standard} = \frac{\lambda O_i - 3\sqrt{\lambda O_i}}{O_i} = \lambda - 3\sqrt{\frac{\lambda}{O_i}}$$

$$= 0 \text{ for } \lambda \le 9\frac{1}{O_i}.$$

Similarly, the adjusted control limits for the rates R_i are:

$$UCL_{Rates\text{-}Adjusted} = \frac{\lambda O_i + 2.782\sqrt{\lambda O_i} + 1}{O_i} = \lambda + 2.782\sqrt{\frac{\lambda}{O_i}} + \frac{1}{O_i}$$

$$LCL_{Rates\text{-}Adjusted} = \frac{\lambda O_i - 2.782\sqrt{\lambda O_i} + 1.1}{O_i} = \lambda - 2.782\sqrt{\frac{\lambda}{O_i}} + \frac{1.1}{O_i}$$

$$= 0 \text{ for } \lambda \le 5.313\frac{1}{O_i}.$$

5.0 Conclusions

U charts are useful for count data following the Poisson distribution, which is most likely for counts below 25. For Poisson data, the U chart is better than an I_N chart because it is based on a better estimate of the standard deviation. However, a U chart is not entirely based on the Poisson distribution because the Poisson distribution is skewed while a U chart uses symmetrical control limits.

For the standard upper control limit, the rate of false detection can be as high as 1 in 11.5 for low counts. It does not remain above 1 in 200 until the average count reaches 10. The standard upper control limit should not be used due to false rejections when the average is less than 10. This can result in wasted resources investigating false signals.

The adjusted UCL control limit fixes this problem. It maintains the false detection rate above 1 in 200 and averages around 1 in 370. The adjusted UCL also improves the detection of a worsening of quality for larger counts, as long as the assumption of the Poisson continues to hold.

For the standard lower control limit, the rate of false detection is consistently above 1 in 1000. This makes it slow to detect improvements in quality. This can result in missing improvements and the associated lessons learned. The adjusted LCL makes the chart much quicker at detecting improvements in quality.

The adjusted control limits result in a U chart truly based on the assumption of the Poisson distribution. The same approach can be applied to P charts as described in Taylor (2017a).

6.0 References

Adjusted Control Limits for P Charts (2017a), Dr. Wayne A. Taylor, Taylor Enterprises, Inc. (Variation.com/adjusted-control-limits-for-p-charts/).

Normalized Individuals (I_N) Chart (2017b), Dr. Wayne A. Taylor, Taylor Enterprises, Inc. (Variation.com/normalized-individuals-control-chart/).

Statistical Procedures for the Medical Device Industry (2017c), Dr. Wayne A. Taylor, Taylor Enterprises, Inc. (Variation.com/procedures).